ISOLATION OF THE SINGULARITY IN THE SOLUTION OF A HEAT CONDUCTION PROBLEM

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A finite-difference method is used to solve a problem with a singularity in the solution. A scheme suitable for describing the behavior of the solution near the singularity is constructed.

In the numerical solution of problems of mathematical physics by finite-difference methods it is usual to make use of the smoothness of the functions considered. Disturbance of smoothness near singular points makes it necessary to use special methods in order to avoid making the mesh interval too fine. One such method, the isolation of singularities, has been discussed by E. A. Volkov in relation to the example of elliptic equations [1, 2].

In this paper we describe a somewhat different approach to the isolation of singularities. We note that the method is not restricted to the problem considered but may be extended to a more general case.

We will consider the problem of heat conduction in a solid bathed on one side by a laminar flow, it being assumed that a homogeneous condition of the second kind is satisfied on the other boundaries adjacent to the flow surface. The simplest of such problems has the following form:

$$c_s \rho s \frac{\partial T_s}{\partial \tau} = \lambda_s \left(\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2} \right) + Q(x, y, \tau), \tag{1}$$

$$(0 < x < d;\ 0 < y < +R;\ \tau > 0);$$

$$T_s|_{\tau=0} = 0; \tag{2}$$

$$\left.\frac{\partial T_s}{\partial x}\right|_{x=0} = 0; \tag{3}$$

$$\left.\frac{\partial T_s}{\partial x}\right|_{x=d} = 0; \tag{4}$$

$$\lambda_s \left.\frac{\partial T_s}{\partial y}\right|_{y=R} = -q(x) \exp(-\beta\tau) - \alpha T_s|_{y=R}, \tag{5}$$

$$c_f \rho_f \left(\frac{\partial T_f}{\partial \tau} + u \frac{\partial T_f}{\partial x} \right) = \lambda_f \frac{\partial^2 T_f}{\partial y^2};\ u = My; \tag{6}$$

$$(0 < x < d,\ -\infty < y < 0,\ \tau > 0);$$

$$T_f|_{\tau=0} = 0; \tag{7}$$

$$T_f|_{x=0} = T_f^{(0)}; \tag{8}$$

$$T_f|_{y=-0} = T_s|_{y=+0}; \tag{9}$$

$$\lambda_f \left.\frac{\partial T_f}{\partial y}\right|_{y=-0} = \lambda_s \left.\frac{\partial T_s}{\partial y}\right|_{y=+0}. \tag{10}$$

In what follows dimensionless quantities will be used exclusively.

We will investigate the singularity occurring close to $\bar{x} = 0$, $\bar{y} = 0$ when the behavior of the solution with respect to $\bar{\tau}$ is sufficiently slow and smooth and motion does not have an important influence on the behavior of the solution with respect to $(\bar{x}, \bar{y})$.

Moreover, for simplicity we assume that $Q(\bar{x}, \bar{y}) \equiv \equiv 0$ in a certain neighborhood of the point $(0, 0)$.

Introducing polar coordinates in the body $(\bar{y} > 0)$ and using a Mellin transformation with respect to r to describe the relation between the temperature of the surface y = 0 and the heat flow through it (since our treatment is a local one, we can confine ourselves to the region $0 \le \bar{r} \le \bar{r}_0$, specifying some boundary condition at $\bar{r} = \bar{r}_0$), and also applying Mellin transformation with respect to $\bar{x}$ to the relation

$$T_f(\bar{x})|_{\bar{y}=0} = N \int_0^{\bar{x}} (\bar{x} - \bar{\xi})^{-2/3} \left.\frac{\partial T(\bar{\xi}, \bar{y})}{\partial \bar{y}}\right|_{\bar{y}=0} d\bar{\xi} \quad (N = 0.512039 K_2),$$

we establish that in the body close to (0, 0) the solution has an asymptotic form (Table 1)

$$T_s = \sum_{n,k} \bar{c}_{n,k} \operatorname{Re}\left(\bar{z}^{n/3} \frac{\ln^k \bar{z}}{3^k} \right). \tag{11}$$

As distinct from [2] we will construct schemes with low connectivity, whose coefficients are found

Table 1

Coefficients $\bar{c}_{n,k}$ *)

Values of k	Values of $\bar{c}_{n,k}$ at n = 0	1	2	3	4	5	6	7	8
0	$\bar{c}_{0,0}$	0	$\frac{0.932478}{K_2}\bar{c}_{0,0}$	0	$\frac{0.297575}{K_2^2}\bar{c}_{0,0}$	0	$\bar{c}_{6,0}$	0	$\frac{0.419615}{K_2}\bar{c}_{6,0} - \frac{0.0179552}{K_2^4}\bar{c}_{0,0}$
1	0	0	0	0	0	0	$\frac{0.123326}{K_2^3}\bar{c}_{0,0}$	0	$\frac{0.0517492}{K_2^4}\bar{c}_{0,0}$

*) The coefficients $\bar{c}_{6i,0}$ are arbitrary.

Table 2

Coefficients of difference scheme $A_{i,j}$ $(B_{i,j} = 2 - A_{i,j})$

Values of j	Values of $A_{i,j}$ at i = 0	1	2	3	4
0	—2	0.612752	0.934496	0.971136	0.983786
1	1.3872	0.7839	0.5514	1.1643	1.0226
2	2.0655	1.0009	0.9407	0.9617	0.8536

from the condition that these equations must be satisfied by the asymptotic solution obtained.

Naturally, the number of parameters in each of the difference equations must correspond with the number of terms retained in the asymptotic expansion. In addition, certain requirements ensuring the stability of the difference scheme must also be satisfied, the description of the solution under other conditions must be at least qualitatively correct, etc. When the number of parameters in the difference equation and the number of terms retained in the asymptotic expansion correspond exactly, all these requirements will not always be satisfied and should therefore be imposed in advance in the form of equalities or inequalities, which may be added, where necessary, to the asymptotic conditions. In this case it may be necessary to increase the connectivity of the difference scheme.

On the other hand, if there is a possibility of estimating the error, it is possible to require its minimization in a suitable class of functions, which may sometimes lead to sufficiently good results even in the case of low connectivity.

For Eq. (1) with conditions (2)–(5) we shall find schemes on a square net ($\bar{x}_i = i\bar{h}_s$, $\bar{y}_i = j\bar{h}_s$), which would be satisfied by asymptotic expression (11). We shall confine ourselves to five-point schemes in which the nonzero coefficients of values of T_{ij} are nonzero at not more than five points, like the simplest approximation of the Laplace operator. If the problem is solved for some fixed value of K_2, it is necessary to find the scheme for which the solutions are determined in accordance with (11) with $\bar{c}_{0,0} = 1$, $\bar{c}_{6,0} = 0$; $\bar{c}_{0,0} = 0$, $\bar{c}_{6,0} = 1$, etc. However, if it is required that the scheme be satisfied at arbitrary K_2 from some region, it is necessary that any term of (11), however remote, be an exact solution. In order to retain the qualitative characteristics of ordinary elliptic net equations it is required that the scheme also be exact for the functions $\bar{x}$, $\bar{y}$ and $\bar{x}^2 + \bar{y}^2$ as well as the first two terms of (11). Thus, for the coefficients of a scheme of the form

$$a_{i,j}T_{i-1,j} + b_{i,j}T_{i,j} + c_{i,j}T_{i+1,j} + d_{i,j}T_{i,j-1} + e_{i,j}T_{i,j+1} = g_{i,j} \quad (12)$$

we obtain the equations

$$a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j} + e_{i,j} = 0,$$

$$h_s\left[(i-1)a_{i,j} + ib_{i,j} + (i+1)c_{i,j} + id_{i,j} + ie_{i,j}\right] = 0,$$

$$h_s\left[ja_{i,j} + jb_{i,j} + jc_{i,j} + (j-1)d_{i,j} + (j+1)e_{i,j}\right] = 0,$$

$$h_s^{2/3}\Big\{a_{i,j}\left[(i-1)^2 + j^2\right]^{1/3}\cos\frac{2}{3}\operatorname{arc\,ctg}\frac{j}{i-1} + b_{i,j}(i^2+j^2)^{1/3}\cos\frac{2}{3}\operatorname{arc\,ctg}\frac{j}{i} + c_{i,j}\left[(i+1)^2 + j^2\right]^{1/3}\cos\frac{2}{3}\times\operatorname{arc\,ctg}\frac{j}{i+1} + d_{i,j}\left[i^2 + (j-1)^2\right]^{1/3}\cos\frac{2}{3}\operatorname{arc\,ctg}\frac{j-1}{i} + e_{i,j}\left[i^2 + (j+1)^2\right]^{1/3}\cos\frac{2}{3}\operatorname{arc\,ctg}\frac{j+1}{i}\Big\} = 0,$$

$$h_s^2\Big\{a_{i,j}\left[(i-1)^2 + j^2\right] + b_{i,j}(i^2+j^2) + c_{i,j}\left[(i+1)^2 + j^2\right] + d_{i,j}\left[i^2 + (j-1)^2\right] + e_{i,j}\left[i^2 + (j+1)^2\right]\Big\} = 4, \quad (13)$$

which give

$$a_{i,j} = c_{i,j} = A_{i,j}/2h_s^2,$$
$$d_{i,j} = e_{i,j} = B_{i,j}/2h_s^2,$$
$$-h_s^2 b_{i,j} = 2(A_{i,j} + B_{i,j}) = 4, \quad (14)$$

or in dimensionless form

$$\bar{A}_{i,j} = 2\Big\{[i^2 + (j+1)^2]^{1/3}\cos\frac{2}{3}\varphi_{i,j+1} - 2(i^2+j^2)^{1/3}\cos\frac{2}{3}\varphi_{i,j} + [i^2 + (j-1)^2]^{1/3}\cos\frac{2}{3}\varphi_{i,j-1}\Big\}\Big\{[i^2 + (j+1)^2]^{1/3}\cos\frac{2}{3}\varphi_{i,j+1} - [(i+1)^2 + j^2]^{1/3}\cos\frac{2}{3}\varphi_{i+1,j} + [i^2 + (j-1)^2]^{1/3}\times\cos\frac{2}{3}\varphi_{i,j-1} - [(i-1)^2 + j^2]^{1/3}\cos\frac{2}{3}\varphi_{i-1,j}\Big\}^{-1}, \quad (15)$$

where

$$\varphi_{i,j} = \operatorname{arcctg}\frac{j}{i}.$$

In this case as the boundary condition at $\bar{x} = 0$ we can take $T_{-1,j} = T_{1,j}$ and substitute in (15) with i = 0, excluding $T_{-1,j}$ from consideration. Close to the line

$$\frac{1}{2}\left(\frac{\bar{x}}{\bar{y}} - \frac{\bar{y}}{\bar{x}}\right) = \operatorname{tg}\frac{2}{3}\operatorname{arctg}\frac{\bar{x}}{\bar{y}},$$

which gives

$$\frac{\bar{x}}{\bar{y}} = \operatorname{tg}\frac{3\pi}{8} = 2.41355,$$

in which relation (15) may lead to a large difference between $\bar{A}_{i,j}$ and $\bar{B}_{i,j}$ and even to their having different signs, which distorts the qualitative description of the process. However, this is due to the fact that the second differences of the function $r^{-2/3}\cdot\cos(2/3)\varphi$ close to this line are small and, consequently, where the deviation of $\bar{A}_{i,j}$ and $\bar{B}_{i,j}$ from unity is large at not very small $\bar{r}$, it is possible to replace them by unity without a serious increase in the error. The results of calculating $\bar{A}_{i,j}$ from (15) are presented in Table 2.

Since the coincidence of the signs of $\bar{A}$ and $\bar{B}$ is disturbed at the point (0, 0), we will not construct a five-point equation for it, but replace $T^S_{0,0}$ with $T^S_{0,1}$ or $T^S_{1,0}$, using the first terms of (11). In the equations containing $T^S_{0,0}$ it is eliminated by means of this expression, which involves a change in the structure of scheme (12).

We will confine ourselves to the case when the behavior of the solution is exponential in time (such solutions can subsequently be used for describing processes of a more general type) $T(\bar{x}, \bar{y}, \bar{\tau}) = T(\bar{x}, \bar{y}) \exp(-\beta, \tau)$. Then it is easy to see that in the body all the basic calculations involving the asymptotic expression will be correct for terms up to $\bar{c}_{4,0}$ if it is possible to neglect the motion in the fluid.

In order to examine the process in the fluid it is convenient to introduce the self-similar coordinate system [3]

$$\bar{\eta} = \bar{x}; \quad \bar{\xi} = \bar{y}/\bar{x}^{-1/3}. \tag{16}$$

Equation (6) takes the form

$$-\bar{\xi}\bar{\eta}\frac{\partial T_f}{\partial \bar{\eta}} = \frac{\partial^2 T_f}{\partial \bar{\xi}^2} - \frac{1}{3}\bar{\xi}^2\frac{\partial T_f}{\partial \bar{\xi}} + K_1\bar{\eta}^{-2/3}T_f$$

$$\left(0 < \bar{\eta} < 1, \quad -\infty < \bar{\xi} < 0\right). \tag{17}$$

In order to construct a scheme for parabolic equation (17), using the ideas of [3], we require that it be exact for solutions of the form

$$\varphi_0(\bar{\xi}); \quad \bar{\eta}^{-2/3}\varphi_{-2/3}(\bar{\xi}); \quad \bar{\eta}^{-2/3}\varphi_{2/3}(\bar{\xi}), \quad \text{where}$$

$$L_{\bar{\xi}}\,\varphi_i(\bar{\xi}) = i\,\bar{\xi}\,\varphi_i(\bar{\xi}); \quad L_{\bar{\xi}} = \frac{\partial^2}{\partial \bar{\xi}^2} - \frac{1}{3}\bar{\xi}^2\frac{\partial}{\partial \bar{\xi}}$$

with the boundary condition $\varphi_i(-\infty) = 0$. As a result, for (17) we obtain the scheme

$$\frac{\bar{\xi}\bar{\eta}_{av}}{h_{\bar{\eta}}}\left[T_f(\bar{\eta}_{i+1}) - T_f(\bar{\eta}_i)\right] = L_{\bar{\xi}}\left[\gamma(\bar{\eta}_i)T_f(\bar{\eta}_i) + \right.$$

$$\left. + \left(1-\gamma(\bar{\eta}_i)\right)T_f(\bar{\eta}_{i+1})\right], \tag{18}$$

where

$$\gamma(\bar{\eta}_i) \equiv \frac{1}{2}; \quad \bar{\eta}_{av}(\bar{\eta}_i) =$$

$$= \frac{h_{\bar{\eta}}}{3}\left(\frac{\bar{\eta}_{i+1}^{-2/3}}{\bar{\eta}_i^{-2/3} - \bar{\eta}_{i+1}^{-2/3}} - \frac{\bar{\eta}_{i+1}^{2/3}}{\bar{\eta}_i^{2/3} - \bar{\eta}_{i+1}^{2/3}}\right);$$

$L_{\bar{\xi}}$ is the ordinary difference approximation of the operator L with interval $h_{\bar{\xi}}$.

The addition to $L_{\bar{\xi}}$ in Eq. (17) is represented in the form

$$K_1\frac{\bar{\eta}_i^{-2/3}T_f(\bar{\eta}_i) + \bar{\eta}_{i+1}^{-2/3}T_f(\bar{\eta}_{i+1})}{2}.$$

To satisfy the conditions (9) and (10) we proceed as follows. We extrapolate T_f to $\bar{\xi} = +h_{\bar{\xi}}$, T_S to $\bar{y} = -h_{\bar{y}}$ on the line $\bar{x} = \bar{\eta} = ih_{\bar{x}} = ih_{\bar{\eta}}$; we represent T_S and T_f, respectively, in the form

$$T_s = m_0 + m_1\bar{r}^{-2/3}\cos\frac{2}{3}\varphi, \tag{18a}$$

$$T_f = m_2 + m_3\bar{\xi} + m_4\bar{\xi}^2. \tag{18b}$$

(The coefficients m_j are different for each $\bar{x} = \bar{\eta}$). We require the satisfaction of (18a) at the point $\bar{y} = \pm h_{\bar{y}}$ and (18b) at the point $\xi = 0$ and $\bar{\xi} = \pm h_{\bar{\xi}}$. To these five equations we add Eq. (12) for (i, 0) and the relation

$$\left(L_{\bar{\xi}}T_f + K_1\bar{\eta}_i^{-2/3}T_f\right)\Big|_{\substack{\bar{\eta}=ih_{\bar{\eta}}\\ \bar{\xi}=0}} = 0.$$

We also add the two requirements on the coefficients m_j following from (9) and (10). Eliminating from these relations m_j, $T^f_{i,0}$, $T^f_{i,+1}$, $T^s_{i,-1}$, we obtain the five-point equation

$$\frac{1}{K_2\bar{D}_i h_{\bar{\xi}}\bar{\eta}_i^{-1/3}}T^f_{i,-1} + \left[\frac{K_1 h_{\bar{\xi}}\bar{\eta}_i^{-1/3}}{2K_2\bar{D}_i} - \frac{1}{K_2\bar{D}_i h_{\bar{\xi}}\bar{\eta}_i^{-1/3}} - \right.$$

$$\left. - \frac{1}{\bar{B}_{i,0}}\left(\beta\bar{h}_s^2 - 2\bar{A}_{i,0}\right) + 2\right]T^s_{i,0} + \left(\frac{\bar{C}_i}{\bar{D}_i} - 1\right)T^s_{i,+1} -$$

$$- \frac{\bar{A}_{i,0}}{\bar{B}_{i,0}}\left(T^s_{i-1,0} + T^s_{i+1,0}\right) = 0, \tag{19}$$

$$\bar{C}_i = \frac{1}{\bar{h}_s}\left(-\frac{\sqrt{3}}{3}i^{-1/3}\right) \times$$

$$\times\left[(i^2+1)^{1/3}\left(\cos\frac{2}{3}\varphi_{i,-1} - \cos\frac{2}{3}\varphi_{i,+1}\right)\right]^{-1}, \tag{20}$$

$$\bar{D}_i = \frac{1}{\bar{h}_s}\left(\frac{\sqrt{3}}{3}i^{-1/3}\right)\times$$

$$\times\left[(i^2+1)^{1/3}\left(\cos\frac{2}{3}\varphi_{i,-1} - \cos\frac{2}{3}\varphi_{i,+1}\right)\right]^{-1}$$

$$\left(\bar{h}_s = h_{\bar{x}} = h_{\bar{y}}\right) \tag{20a}$$

for $i \geq 1$, it being necessary to eliminate $T^S_{0,0}$ at i = 1. It is not necessary to calculate $T^S_{0,0}$ and $T_f\big|_{\bar{\xi}=0}$ in the process of solving the finite-difference problem. This is necessary only for the subsequent investigation of the transfer process. We note that remote from the point (0, 0) it is possible to use ordinary methods for approximating the differential equations. An actual application of the schemes developed above will be described in connection with the investigation of a boundary-value transport problem.

NOTATION

c_S, ρ_S, λ_S, a_S, c_f, ρ_f, λ_f, a_f are the specific heat, density and the coefficients of thermal conductivity and thermal diffusivity of solid and fluid, respectively; T_S, T_f are the temperatures of the solid and fluid; d is the longitudinal dimension of body; R is the transverse dimension of the body, u is the fluid velocity; M is a constant; $\bar{x} = x/d$, $\bar{y} = y/d$ are dimensionless variable coordinates in the body; $\bar{\tau} = \frac{a_s}{d^2}\tau$ is dimensionless time; $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$; $\bar{\eta} = \bar{x}$, $\bar{\xi} = \bar{y}/\bar{x}^{-1/3}$ are self-similar dimensionless coordinates in the fluid; $\bar{\beta} = \frac{d^2}{}\beta$ is the dimensionless relative cooling rate; $K_1 = a_s\bar{\beta}/M^{2/3}a_f^{1/3}d^{4/3}$,

$K_2 = c_s \rho_s a_s / c_f \rho_f a_f^{2/3} M^{1/3} d^{2/3}$ are the dimensionless complexes; i is the numbering of nodes of finite-difference scheme in the body with respect to $\bar{x}$ and in the fluid with respect to $\bar{\eta}$; j is the numbering of nodes of net in the body with respect to $\bar{y}$ and in the fluid with respect to $\bar{\xi}$; $h_{\bar{x}} = h_{\bar{y}} = \bar{h}_s$ is the dimensionless mesh interval in the body; $h_{\bar{\eta}}$, $h_{\bar{\xi}}$ are the dimensionless intervals with respect to $\bar{\eta}$ and $\bar{\xi}$ in the fluid; $\bar{z} = \bar{x} + \sqrt{-1}\,\bar{y}$.

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3 August 1966

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